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COMPACTNESS IN L-FUZZY TOPOLOGICAL SPACES

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Abstract. We give a definition of compactness in L-fuzzy topological spaces and provide a characterization of compact L-fuzzy topological spaces, where L is a complete quasi-monoidal lattice with some additional structures, and we present a version of Tychonoff's theorem within the category of L-fuzzy topological spaces.

0. Introduction

Over the years, a number of descriptions of compactness of fuzzy topological spaces has appeared. As mathematicians sought to general topology in various ways using the concept of fuzzy subsets of an ordinary set, it is not surprising that searches for such properties were obtained with different degrees of success, depending on the structure of the underlying lattice L. One of the main goals of any theory of compactness is to investigate the problem to which extent the formulation of a theorem of Tychonoff is possible. The aim of this paper is to present a version of Tychonoff's theorem within the category of L-fuzzy topological spaces, when the underlying lattice L is a cqm-lattice with some additional structures. Following P. T. Johnstone ([10]), within the text of the paper, those propositions, lemmas and theorems whose proofs require Zorn's lemma are distinguished by being marked with an asterisk.

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The paper is organized as follows: After some lattice-theoretical prerequisites, where we briefly recall the concept of a cqm-lattice, in section 2 we shall present the concept of compact L-fuzzy topological spaces. Finally, in section 3 we present a proof of the Tychonoff's theorem.

1. From Lattice Theoretic Foundations

Let $(L, \leq)$ be a complete, infinitely distributive lattice, i.e. $(L, \leq)$ is a partially ordered set such that for every subset $A \subset L$ the join $\bigvee A$ and the meet $\bigwedge A$ are defined, moreover $(\bigvee A) \wedge \alpha = \bigvee\{a \wedge \alpha) \mid a \in A\}$ and $(\bigwedge A) \vee \alpha = \bigwedge\{a \vee \alpha) \mid a \in A\}$ for every $\alpha \in L$. In particular, $\bigvee L =: \top$ and $\bigwedge L =: \bot$ are respectively the universal upper and the universal lower bounds in L. We assume that $\bot \neq \top$, i.e. L has at least two elements.

1.1. **c**$qm-$**lattices.** The definition of complete quasi-monoidal lattices introduced by E. Rodabaugh in [11] is the following:

A $cqm-$lattice (short for complete quasi-monoidal lattice) is a triple $(l, \leqslant, \otimes)$ provided with the following properties

(1) $(L, \leqslant)$ is a complete lattice with upper bound $\top$ and lower bound $\bot$.
(2) $\otimes : L \times L :\to L$ is a binary operation satisfying the following axioms:
   (a) $\otimes$ is isotone in both arguments, i.e. $\alpha_1 \leqslant \alpha_2$, $\beta_1 \leqslant \beta_2$ implies $\alpha_1 \otimes \beta_1 \leqslant \alpha_2 \otimes \beta_2$;
   (b) $\top$ is idempotent, i.e. $\top \otimes \top = \top$.

1.2. $GL-$**monoids.** A $GL-$monoid (see [6], [7], [8]) is a complete lattice enriched with a further binary operation $\otimes$, i.e. a triple $(L, \leq, \otimes)$ such that:

(1) $\otimes$ is isotone, i.e. $\alpha \leq \beta$ implies $\alpha \otimes \gamma \leq \beta \otimes \gamma$, $\forall \alpha, \beta, \gamma \in L$;
(2) $\otimes$ is commutative, i.e. $\alpha \otimes \beta = \beta \otimes \alpha$, $\forall \alpha, \beta \in L$;

(3) $\otimes$ is associative, i.e. $\alpha \otimes (\beta \otimes \gamma) = (\alpha \otimes \beta) \otimes \gamma$, $\forall \alpha, \beta, \gamma \in L$;

(4) $(L, \leq, \otimes)$ is integral, i.e. $\top$ acts as the unity: $\alpha \otimes \top = \alpha$, $\forall \alpha \in L$;

(5) $\perp$ acts as the zero element in $(L, \leq, \otimes)$, i.e. $\alpha \otimes \perp = \perp$, $\forall \alpha \in L$;

(6) $\otimes$ is distributive over arbitrary joins, i.e. $\alpha \otimes (\bigvee_\lambda \beta_\lambda) = \bigvee_\lambda (\alpha \otimes \beta_\lambda)$, $\forall \alpha \in L, \forall \{\beta_\lambda : \lambda \in I\} \subset L$;

(7) $(L, \leq, \otimes)$ is divisible, i.e. $\alpha \leq \beta$ implies the existence of $\gamma \in L$ such that $\alpha = \beta \otimes \gamma$.

It is well known that every $GL-$monoid is residuated, i.e. there exists a further binary operation "$\longmapsto$" (implication) on L satisfying the following condition:

$$\alpha \otimes \beta \leq \gamma \Longleftrightarrow \alpha \leq (\beta \longmapsto \gamma) \qquad \forall \alpha, \beta, \gamma \in L.$$

Explicitly implication is given by

$$\alpha \longmapsto \beta = \bigvee \{\lambda \in L \mid \alpha \otimes \lambda \leq \beta\}.$$

Important examples of GL-monoids are Heyting algebras and MV-algebras. Namely, a *Heyting algebra* is GL-monoid of the type $(L, \leq, \wedge, \vee, \wedge)$ (i.e. in case of a Heyting algebra $\wedge = \otimes$), cf. e.g. [10]. A GL-monoid is called an *MV-algebra* if $(\alpha \longmapsto \perp) \longmapsto \perp = \alpha \quad \forall \alpha \in L$, [4], [5], see also [8, Lemma 2.14]. Thus in an MV-algebra an order reversing involution $^c : L \to L$ can be naturally defined by setting $\alpha^c := \alpha \longmapsto \perp \quad \forall \alpha \in L$.

If X is a set and L is a GL-monoid, then the fuzzy powerset L^X in an obvious way can be pointwise endowed with a structure of a GL-monoid. In particular the L-sets 1_X and 0_X defined by $1_X(x) := \top$ and $0_X(x) := \perp$ $\forall x \in X$ are respectively the universal upper and lower bounds in L^X.

1.3. **Co-GL-monoids.** A co-$GL-$monoid is a complete lattice with a further binary operation $\oplus$, i.e. a triple $(L, \leqslant, \oplus)$ such that:

(1) $\oplus$ is isotone, i.e. $\alpha \leqslant \beta$ implies $\alpha \oplus \gamma \leqslant \beta \oplus \gamma$, $\forall \alpha, \beta, \gamma \in L$;

(2) $\oplus$ is commutative, i.e. $\alpha \oplus \beta = \beta \oplus \alpha$, $\forall \alpha, \beta \in L$;

(3) $\oplus$ is associative, i.e. $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$, $\forall \alpha, \beta, \gamma \in L$;

(4) $(L, \leq, \oplus)$ is co-integral, i.e. $\perp$ acts as the unity: $\alpha \oplus \perp = \alpha$, $\forall \alpha \in L$;

(5) $\top$ acts as the co-zero element in $(L, \leqslant, \oplus)$, i.e. $\alpha \oplus \top = \top$, $\forall \alpha \in L$;

(6) $\oplus$ is distributive over arbitrary meets, i.e. $\alpha \oplus (\bigwedge_\lambda \beta_\lambda) = \bigwedge_\lambda (\alpha \oplus \beta_\lambda)$, $\forall \alpha \in L, \forall \{\beta_\lambda : \lambda \in I\} \subset L$;

(7) $(L, \leqslant, \oplus)$ is co-divisible, i.e. $\alpha \leqslant \beta$ implies the existence of $\gamma \in L$ such that $\alpha \oplus \gamma = \beta$.

Every co-$GL-$monoid is co-residuated, i.e. there exists a further binary operation "$\leftarrowtail$" (co-implication) on L satisfying the following condition:

$$\alpha \leftarrowtail \beta \leqslant \gamma \Longleftrightarrow \alpha \leqslant (\beta \oplus \gamma) \qquad \forall \alpha, \beta, \gamma \in L.$$

Explicitly, coimplication is given by

$$\alpha \leftarrowtail \beta = \bigwedge \{\lambda \in L \mid \alpha \leqslant \beta \oplus \lambda\}.$$

If X is a set and L is a co-GL-monoid, then the fuzzy powerset L^X in an obvious way can be pointwise endowed with a structure of a co-GL-monoid.

Remark 1.1. *In this paper we will use the particular case in which* $\oplus = \vee$ *and the co-implication is*

$$\alpha \rhd \beta = \bigwedge \{\lambda \in L \mid \alpha \leqslant \beta \vee \lambda\}.$$

In the sequel $(L, \leqslant, \otimes)$ denotes an arbitrary GL-monoid and $(L, \leqslant, \vee)$ denotes an co-GL-monoid.

2. Compactness and L-fuzzy Topological Spaces

In this section we recall some topological concepts (as in [9]) from the theory of L-fuzzy topological spaces.

Definition 2.1. *Given* $(L, \leqslant, \otimes)$ *a cqm-lattice and* X *a non-empty set; an* L*-fuzzy topology on* X *is a map* $\tau : L^X \to L$ *satisfying the following axioms:*

o1. $\tau(1_X) = \top$,

o2. *For* $f, g \in L^X$, $\tau(f) \otimes \tau(g) \leqslant \tau(f \otimes g)$,

o3. *For every subset* $\{f_\lambda\}_{\lambda \in I}$ *of* L^X *the inequality*

$$\bigwedge_{\lambda \in I} \tau(f_\lambda) \leqslant \tau(\bigvee_{\lambda \in I} f_\lambda)$$

holds.

If τ *is an* L*-fuzzy topology on* X, *the pair* (X, τ) *is an* L*-fuzzy topological space.*

If the set X is non-empty, then L^X consists at least of two elements. In particular, the universal lower bound in $(L^X, \leqslant)$ is given by 0_X. When we take the empty subset of L^X and we apply the axiom o3, we obtain

o1'. $\tau(0_X) = \top$.

Definition 2.2. *Given* (X, τ) *and* (Y, η) L*-fuzzy topological spaces, a map* $\phi : X \to Y$ *is* L*-fuzzy continuous iff for all* $g \in L^Y$, ϕ *satisfies*

$$\eta(g) \leqslant \tau(g \circ \phi).$$

In this way, we have the category **L-FTop** where the objects are the L-fuzzy topological spaces and the morphisms are the L-fuzzy continuous maps.

On the set $\mathfrak{T}_L(X)$ of all L-fuzzy topologies on X we have a partial ordering:

$$\tau_1 \leqslant \tau_2 \quad \Leftrightarrow \quad \tau_1(f) \leqslant \tau_2(f), \quad \forall f \in L^X.$$

Remark 2.3. *On the set* $L^X \times L$ *we introduce a partial ordering by*

$$(f, \alpha) \preccurlyeq (g, \beta) \quad \Leftrightarrow \quad (f \leqslant g \ \text{ and } \ \beta \leqslant \alpha)$$

Lemma 2.4. *Let L be a complete lattice such that $(L, \leqslant, \otimes)$ is a $GL-$monoid and $(L, \leqslant, \vee)$ is a co-$GL-$monoid, then $(L^X \times L, \preccurlyeq, \boxtimes)$, where*

$$(f, \alpha) \boxtimes (g, \beta) := (f \otimes g, \alpha \vee \beta), \qquad \forall (f, \alpha), (g, \beta) \in L^X \times L,$$

is a $GL-$monoid.

Proof. It is straightforward to check that for $\{(f_\lambda, \alpha_\lambda)\}_{\lambda \in I} \subseteq L^X \times L$:

(i) $\bigwedge_{\lambda \in I}(f_\lambda, \alpha_\lambda) = \left(\bigwedge_{\lambda\, \in\, I} f_\lambda, \bigvee_{\lambda\, \in\, I} \alpha_\lambda\right)$.

(ii) $\bigvee_{\lambda \in I}(f_\lambda, \alpha_\lambda) = \left(\bigvee_{\lambda\, \in\, I} f_\lambda, \bigwedge_{\lambda\, \in\, I} \alpha_\lambda\right)$.

(iii) $\top = (1_X, \bot)$ is the universal upper bound of $(L, \leqslant, \otimes)$.

(iv) $\bot = (0_X, \top)$ is the universal lower bound of $(L, \leqslant, \otimes)$.

Consequently, $\boxtimes$ is isotone, commutative and associative; $(L, \leqslant, \otimes)$ is integral with respect to $\top$ and $\bot$ acts as the zero element in $(L, \leqslant, \otimes)$.

(v) $\boxtimes$ is distributive over arbitrary joins, i.e. for all $(f, \alpha) \in L^X \times L$, for all $\{(g_j, \beta_j) \mid j \in I\} \subseteq L^X \times L$,

$$\begin{aligned}
(f, \alpha) \boxtimes \left[\bigvee_{j \in I}(g_j, \beta_j)\right] &= (f, \alpha) \boxtimes \left(\bigvee_{j \in I} g_j, \bigwedge_{j \in I} \beta_j\right) \\
&= \left(f \otimes \left(\bigvee_{j \in I} g_j\right), \alpha \vee \left(\bigwedge_{j \in I} \beta_j\right)\right) \\
&= \left(\bigvee_{j \in I}(f \otimes g_j), \bigwedge_{j \in I}(\alpha \vee \beta_j)\right) \\
&= \bigvee_{j \in I} (f \otimes g_j, \alpha \vee \beta_j) \\
&= \bigvee_{j \in I} \left[(f, \alpha) \boxtimes (g_j, \beta_j)\right].
\end{aligned}$$

(vi) $(L^X \times L, \preccurlyeq, \boxtimes)$ is divisible, i.e. $(f, \alpha) \preccurlyeq (g, \beta) \Leftrightarrow f \leqslant g$ and $\beta \leqslant \alpha$ implies the existence of $h \in L^X$ and $\gamma \in L$ such that $f = g \otimes h$ and $\beta \vee \gamma = \alpha$, in other words $(f, \alpha) = (g \otimes h, \beta \vee \gamma) = (g, \beta) \boxtimes (h, \gamma)$. $\square$

Remark 2.5. *The residuation in* $(L^X \times L, \preccurlyeq, \boxtimes)$ *is the binary operation* $\longmapsto$ *given by*

$$\begin{aligned}(f,\alpha) \longmapsto (g,\beta) &= \bigvee_{h\in L^X} \bigwedge_{\eta \in L} \{(h,\eta) \mid h\otimes f \leqslant g, \;\; \beta \leqslant \eta \vee \alpha\} \\ &= (f \longmapsto g, \beta \rhd \alpha),\end{aligned}$$

which satisfies the following condition

$$(f,\alpha) \boxtimes (g,\beta) \preccurlyeq (h,\gamma) \Leftrightarrow (f,\alpha) \preccurlyeq [(g,\beta) \longmapsto (h,\gamma)]\,,$$

$\forall (f,\alpha),(g,\beta),(h,\gamma) \in L^X \times L$. *It is also straightforward to check that*

- $(f,\alpha) \boxtimes [(g,\beta) \longmapsto (h,\gamma)] \preccurlyeq [(g,\beta) \longmapsto (f,\alpha) \boxtimes (h,\gamma)]\,.$
- $[(g,\beta) \longmapsto (f,\alpha)] \boxtimes [(g,\beta) \longmapsto (h,\gamma)] \preccurlyeq [(g,\beta) \longmapsto (f,\alpha) \boxtimes (h,\gamma)]\,,$ *whenever* $\otimes$ *will be idempotent.*

Definition 2.6 (**L-fuzzy filter**)**.** *Let X be a set. A map $\mathcal{F} : L^X \times L \to L$ is called an L-fuzzy filter on X if and only if $\mathcal{F}$ satisfies the following axioms:*

(FF0) $\mathcal{F}(1_X,\alpha) = \top$.

(FF1) $(f,\alpha) \preccurlyeq (g,\beta) \Rightarrow \mathcal{F}(f,\alpha) \leqslant \mathcal{F}(g,\beta)$.

(FF2) $\mathcal{F}(f,\alpha) \otimes \mathcal{F}(g,\beta) \leqslant \mathcal{F}(f\otimes g, \alpha \vee \beta)$.

(FF3) $\mathcal{F}(0_X,\alpha) = \bot$.

Let $\mathfrak{F}_{LF}(X)$ be the set of all L-fuzzy filters on X. On $\mathfrak{F}_{LF}(X)$ we introduce a partial ordering $\preccurlyeq$ by

$$\mathcal{F}_1 \preccurlyeq \mathcal{F}_2 \Leftrightarrow \mathcal{F}_1(f,\alpha) \leqslant \mathcal{F}_2(f,\alpha), \quad \forall (f,\alpha) \in L^X \times L$$

***Proposition 2.7.** *The partially ordered set* $(\mathfrak{F}_{LF}(X), \preccurlyeq)$ *has maximal elements.*

Proof. Referring to Zorn's lemma, it is sufficient to show that every chain $\mathcal{C}$ in $\mathfrak{F}_{LF}(X)$ has an upper bound in $\mathfrak{F}_{LF}(X)$. For this purpose let us consider a non-empty chain $\mathcal{C} = \{\mathcal{F}_\lambda \mid \lambda \in I\}$. We define a map $\mathcal{F}_\infty : L^X \times L \to L$ by

$$\mathcal{F}_\infty(f,\alpha) = \bigvee_{\lambda \in I} \mathcal{F}_\lambda(f,\alpha),$$

and we show that $\mathcal{F}_\infty$ is an L-fuzzy filter on X. In fact

(FF0) $\mathcal{F}_\infty(1_X,\alpha) = \bigvee_{\lambda\in I} \mathcal{F}_\lambda(1_X,\alpha) = \bigvee_{\lambda\in I} \top = \top$.

(FF1) $(f,\alpha) \preccurlyeq (g,\beta) \Rightarrow \mathcal{F}_\infty(f,\alpha) = \bigvee_{\lambda\in I} \mathcal{F}_\lambda(f,\alpha) \leqslant \bigvee_{\lambda\in I} \mathcal{F}_\lambda(g,\beta) = \mathcal{F}_\infty(g,\beta)$.

(FF2)

$$\begin{aligned}\mathcal{F}_\infty(f,\alpha) \otimes \mathcal{F}_\infty(g,\beta) &= \left(\bigvee_{\lambda\in I} \mathcal{F}_\lambda(f,\alpha)\right) \otimes \left(\bigvee_{\lambda\in I} \mathcal{F}_\lambda(g,\beta)\right)\\ &= \bigvee_{\lambda\in I} \left[\mathcal{F}_\lambda(f,\alpha) \otimes \mathcal{F}_\lambda(g,\beta)\right]\\ &\leqslant \bigvee_{\lambda\in I} \left[\mathcal{F}_\lambda(f\otimes g,\alpha\vee\beta)\right]\\ &= \mathcal{F}_\infty(f\otimes g,\alpha\vee\beta).\end{aligned}$$

(FF3) $\mathcal{F}_\infty(0_X,\alpha) = \bigvee_{\lambda\in I} \mathcal{F}_\lambda(0_X,\alpha) = \bigvee_{\lambda\in I} \bot = \bot$.

□

Definition 2.8. *A maximal element in* $(\mathfrak{F}_{LF}(X), \preccurlyeq)$ *is also called an L-fuzzy ultrafilter.*

***Proposition 2.9.** *For every L-fuzzy filter* $\mathcal{U} : L^X \times L \to L$ *on* X *the following assertions are equivalent*

(i) $\mathcal{U}$ *is an L-fuzzy ultrafilter.*

(ii) $\mathcal{U}(f,\alpha) = (\mathcal{U}\,[(f,\alpha) \longmapsto (0_X,\rho)]) \longmapsto \bot$, *for all* $(f,\alpha) \in L^X \times L$, *for all* $\rho \leqslant \alpha$ *in* L.

Proof. $(i) \Rightarrow (ii)$

Because of $(FF2)$ and $(FF3)$ every L-fuzzy filter satisfies the condition

(FF3') $\mathcal{U}(f,\alpha) \leqslant (\mathcal{U}\left[(f,\alpha) \longmapsto (0_X,\rho)\right]) \longmapsto \perp$, for all $(f,\alpha) \in L^X \times L$, for all $\rho \leqslant \alpha$ in L.

In order to verify $(i) \Rightarrow (ii)$ it is sufficient to show that the maximality of $\mathcal{U}$ implies

$$(\mathcal{U}\left[(f,\alpha) \longmapsto (0_X,\rho)\right]) \longmapsto \perp \leqslant \mathcal{U}(f,\alpha),\ \forall (f,\alpha) \in L^X \times L,\ \forall \rho \leqslant \alpha \text{ in } L.$$

For this purpose, we fix an element $(g,\beta) \in L^X \times L$, for that element we let $\mathcal{G}_\beta := (\mathcal{U}\left[(g,\beta) \longmapsto (0_X,\rho)\right]) \longmapsto \perp$ and define a map $\hat{\mathcal{U}} : L^X \times L \to L$ by

$$\hat{\mathcal{U}}(f,\alpha) = \mathcal{U}(f,\alpha) \bigvee \Big\{ \mathcal{U}\left[(g,\beta) \longmapsto (f,\alpha)\right] \otimes \mathcal{G}_\beta \Big\}.$$

We must show that $\hat{\mathcal{U}}$ is an L-fuzzy ultrafilter. Firstly $\hat{\mathcal{U}}$ is an L-fuzzy filter: obviously $\hat{\mathcal{U}}$ satisfies $(FF0)$. For the axiom $(FF1)$, from the definition

$$\hat{\mathcal{U}}(f,\alpha) = \mathcal{U}(f,\alpha) \bigvee \Big\{ \mathcal{U}\left[(g,\beta) \longmapsto (f,\alpha)\right] \otimes \mathcal{G}_\beta \Big\}$$

and

$$\hat{\mathcal{U}}(h,\gamma) = \mathcal{U}(h,\gamma) \bigvee \Big\{ \mathcal{U}\left[(g,\beta) \longmapsto (h,\gamma)\right] \otimes \mathcal{G}_\beta \Big\}.$$

Now, for $(f,\alpha) \preccurlyeq (h,\gamma)$ we have that $\mathcal{U}(f,\alpha) \leqslant \mathcal{U}(h,\gamma)$, moreover,

$$\begin{aligned} (g,\beta) \longmapsto (f,\alpha) &= \bigvee \big\{ (k,\delta) \in L^X \times L \mid (g,\beta) \boxtimes (k,\delta) \preccurlyeq (f,\alpha) \big\} \\ &\leqslant \bigvee \big\{ (k,\delta) \in L^X \times L \mid (g,\beta) \boxtimes (k,\delta) \preccurlyeq (h,\gamma) \big\} \\ &= (g,\beta) \longmapsto (h,\gamma), \end{aligned}$$

which implies that

$$\begin{aligned} \hat{\mathcal{U}}(f,\alpha) &= \mathcal{U}(f,\alpha) \bigvee \Big\{ \mathcal{U}\left[(g,\beta) \longmapsto (f,\alpha)\right] \otimes \mathcal{G}_\beta \Big\} \\ &\leqslant \mathcal{U}(h,\gamma) \bigvee \Big\{ \mathcal{U}\left[(g,\beta) \longmapsto (h,\gamma)\right] \otimes \mathcal{G}_\beta \Big\} \\ &= \hat{\mathcal{U}}(h,\gamma). \end{aligned}$$

For the axiom $(FF2)$, we must verify that

$$\hat{\mathcal{U}}(f,\alpha)\otimes\hat{\mathcal{U}}(h,\gamma)\leqslant\hat{\mathcal{U}}(f\otimes h,\alpha\vee\gamma),$$

In fact,

$$\hat{\mathcal{U}}(f,\alpha)\otimes\hat{\mathcal{U}}(h,\gamma)$$

$$=\big(\mathcal{U}(f,\alpha)\bigvee\{\mathcal{U}\left[(g,\beta)\longmapsto(f,\alpha)\right]\otimes\mathcal{G}_\beta\}\big)$$

$$\otimes\big(\mathcal{U}(h,\gamma)\bigvee\{\mathcal{U}\left[(g,\beta)\longmapsto(h,\gamma)\right]\otimes\mathcal{G}_\beta\}\big)$$

$$=\mathcal{U}(f,\alpha)\otimes\Big[\mathcal{U}(h,\gamma)\bigvee\{\mathcal{U}\left[(g,\beta)\longmapsto(h,\gamma)\right]\otimes\mathcal{G}_\beta\}\Big]$$

$$\bigvee\Big[\big\{\mathcal{U}\left[(g,\beta)\longmapsto(f,\alpha)\right]\otimes\mathcal{G}_\beta\big\}\otimes\big\{\mathcal{U}(h,\gamma)\bigvee\{\mathcal{U}\left[(g,\beta)\longmapsto(h,\gamma)\right]\otimes\mathcal{G}_\beta\}\big\}\Big]$$

$$=\mathcal{U}(f,\alpha)\otimes\mathcal{U}(h,\gamma)\bigvee\left[\mathcal{U}(f,\alpha)\otimes\big\{\mathcal{U}\left[(g,\beta)\longmapsto(h,\gamma)\right]\otimes\mathcal{G}_\beta\big\}\right]$$

$$\bigvee\big(\big\{\mathcal{U}\left[(g,\beta)\longmapsto(f,\alpha)\right]\otimes\mathcal{G}_\beta\big\}\otimes\mathcal{U}(h,\gamma)\big)$$

$$\bigvee\big(\big\{\mathcal{U}\left[(g,\beta)\longmapsto(f,\alpha)\right]\otimes\mathcal{G}_\beta\big\}\otimes\big\{\mathcal{U}\left[(g,\beta)\longmapsto(h,\gamma)\right]\otimes\mathcal{G}_\beta\big\}\big)$$

$$=\mathcal{U}(f,\alpha)\otimes\mathcal{U}(h,\gamma)\bigvee\left[\big\{\mathcal{U}(f,\alpha)\otimes\mathcal{U}\left[(g,\beta)\longmapsto(h,\gamma)\right]\big\}\otimes\mathcal{G}_\beta\right]$$

$$\bigvee\big(\big\{\mathcal{U}(h,\gamma)\otimes\mathcal{U}\left[(g,\beta)\longmapsto(f,\alpha)\right]\big\}\otimes\mathcal{G}_\beta\big)$$

$$\bigvee\big(\big\{\mathcal{U}\left[(g,\beta)\longmapsto(f,\alpha)\right]\otimes\mathcal{U}\left[(g,\beta)\longmapsto(h,\gamma)\right]\big\}\otimes\mathcal{G}_\beta\big)$$

$$\leqslant\mathcal{U}(f\otimes h,\alpha\vee\gamma)\bigvee\left[\mathcal{U}\left((f,\alpha)\boxtimes\left[(g,\beta)\longmapsto(h,\gamma)\right]\right)\otimes\mathcal{G}_\beta\right]$$

$$\bigvee\left(\mathcal{U}\left((h,\gamma)\boxtimes\left[(g,\beta)\longmapsto(f,\alpha)\right]\right)\otimes\mathcal{G}_\beta\right)$$

$$\bigvee\left(\mathcal{U}\left(\left[(g,\beta)\longmapsto(f,\alpha)\right]\boxtimes\left[(g,\beta)\longmapsto(h,\gamma)\right]\right)\otimes\mathcal{G}_\beta\right)$$

$$\leqslant\mathcal{U}(f\otimes h,\alpha\vee\gamma)\bigvee\left[\mathcal{U}\left[(g,\beta)\longmapsto(f,\alpha)\boxtimes(h,\gamma)\right]\otimes\mathcal{G}_\beta\right]$$

$$\bigvee\left[\mathcal{U}\left[(g,\beta)\longmapsto(f,\alpha)\boxtimes(h,\gamma)\right]\otimes\mathcal{G}_\beta\right]$$

$$\bigvee\left[\mathcal{U}\left[(g,\beta)\longmapsto(f,\alpha)\boxtimes(h,\gamma)\right]\otimes\mathcal{G}_\beta\right]$$

$$=\mathcal{U}(f\otimes h,\alpha\vee\gamma)\bigvee\left[\mathcal{U}\left[(g,\beta)\longmapsto(f,\alpha)\boxtimes(h,\gamma)\right]\otimes\mathcal{G}_\beta\right]$$

$$=\hat{\mathcal{U}}(f\otimes h,\alpha\vee\gamma).$$

In order to verify $(FF3)$, we have that

$$
\begin{aligned}
\hat{\mathcal{U}}(0_X,\alpha) &= \mathcal{U}(0_X,\alpha) \bigvee \Big\{\mathcal{U}\left[(g,\beta) \longmapsto (0_X,\alpha)\right] \otimes \mathcal{G}_\beta\Big\} \\
&= \perp \vee \Big\{\mathcal{U}\left[(g,\beta) \longmapsto (0_X,\alpha)\right] \otimes \mathcal{G}_\beta\Big\} \\
&= \mathcal{U}\left[(g,\beta) \longmapsto (0_X,\alpha)\right] \otimes \mathcal{G}_\beta.
\end{aligned}
$$

On the other hand, using the fact that $\rho \leqslant \alpha$, we conclude that $\rho \rhd \beta \leqslant \alpha \rhd \beta$ which implies $(g,\beta) \longmapsto (0_X,\alpha) \preccurlyeq (g,\beta) \longmapsto (0_X,\rho)$. Therefore,

$$
\mathcal{U}\left((g,\beta) \longmapsto (0_X,\alpha)\right) \leqslant \mathcal{U}\left((g,\beta) \longmapsto (0_X,\rho)\right),
$$

i.e.

$$
\mathcal{G}_\beta \leqslant \left[\mathcal{U}\left((g,\beta) \longmapsto (0_X,\rho)\right) \longmapsto \perp\right].
$$

Now we invoke the residuation property of $(L, \leqslant, \otimes)$ to obtain

$$
\hat{\mathcal{U}}(0_X,\alpha) = \Big\{\mathcal{U}\left[(g,\beta) \longmapsto (0_X,\alpha)\right] \otimes \mathcal{G}_\beta\Big\} = \perp.
$$

Now we must show that $\hat{\mathcal{U}}$ is an L-fuzzy ultrafilter on X. In fact, since

$$
\hat{\mathcal{U}}(f,\alpha) = \mathcal{U}(f,\alpha) \bigvee \Big\{\mathcal{U}\left[(g,\beta) \longmapsto (f,\alpha)\right] \otimes \mathcal{G}_\beta\Big\},
$$

clearly $\mathcal{U}(f,\alpha) \leqslant \hat{\mathcal{U}}(f,\alpha), \quad \forall (f,\alpha) \in L^X \times L$, but $\mathcal{U}$ is an L-fuzzy ultrafilter on X, therefore $\hat{\mathcal{U}} = \mathcal{U}$. In this way

$$
\begin{aligned}
\mathcal{U}(g,\beta) &= \mathcal{U}(g,\beta) \bigvee \Big\{\mathcal{U}\left[(g,\beta) \longmapsto (g,\beta)\right] \otimes \mathcal{G}_\beta\Big\} \\
&= \mathcal{U}(g,\beta) \bigvee \Big\{\mathcal{U}(1_X,\perp) \otimes \mathcal{G}_\beta\Big\} \\
&= \mathcal{U}(g,\beta) \bigvee \Big\{\top \otimes \mathcal{G}_\beta\Big\} \\
&= \mathcal{U}(g,\beta) \vee \mathcal{G}_\beta.
\end{aligned}
$$

Therefore,

$$
\mathcal{G}_\beta = \left(\mathcal{U}\left[(g,\beta) \longmapsto (0_X,\rho)\right]\right) \longmapsto \perp \leqslant \mathcal{U}(g,\beta), \quad \forall (g,\beta) \in L^X \times L.
$$

From the last inequality and $(FF3')$ we obtain (ii).

$(ii) \Rightarrow (i)$

We must verify that if

$$\mathcal{U}(f,\alpha) = (\mathcal{U}\left[(f,\alpha) \longmapsto (0_X,\rho)\right]) \longmapsto \bot, \text{ for all } (f,\alpha) \in L^X \times L, \text{ for all } \rho \leqslant \alpha \text{ in } L$$

then $\mathcal{U}$ is an L-fuzzy ultrafilter on X.

Suppose $\mathcal{U} \leqslant \hat{\mathcal{U}}$, then

$$\left(\left(\hat{\mathcal{U}}\left[(f,\alpha) \longmapsto (0_X,\rho)\right]\right) \longmapsto \bot\right) \leqslant \left((\mathcal{U}\left[(f,\alpha) \longmapsto (0_X,\rho)\right]) \longmapsto \bot\right),$$

therefore $\hat{\mathcal{U}} \leqslant \mathcal{U}$, consequently $\mathcal{U}$ is an L-fuzzy ultrafilter on X. □

*__Proposition 2.10.__ *Let $\phi : X \to Y$ be a map and let $\mathcal{F} : L^X \times L \to L$ be an L-fuzzy filter on X. Then*

(1) *The map $\phi^{\rightarrow}_{\mathcal{F}} : L^Y \times L \to L$ defined by $\phi^{\rightarrow}_{\mathcal{F}}(g,\beta) = \mathcal{F}(g \circ \phi, \beta)$, $\forall (g,\beta) \in L^Y \times L$ is an L-fuzzy filter on Y.*

(2) *The map $\phi^{\rightarrow}_{\mathcal{U}} : L^Y \times L \to L$ is an L-fuzzy ultrafilter on Y, whenever $\mathcal{U}$ will be an L-fuzzy ultrafilter on X*

Proof. (1). We must show $\phi^{\rightarrow}_{\mathcal{F}}$ satisfies the axioms of an L-fuzzy filter: In fact,

(FF0) $\phi^{\rightarrow}_{\mathcal{F}}(1_Y,\beta) = \mathcal{F}(1_Y \circ \phi, \beta) = \mathcal{F}(1_X,\beta) = \top, \quad \forall \beta \in L.$

(FF1) $(f,\alpha) \preccurlyeq (g,\beta) \Rightarrow \phi^{\rightarrow}_{\mathcal{F}}(f,\alpha) = \mathcal{F}(f \circ \phi, \alpha) \leqslant \mathcal{F}(g \circ \phi, \beta) = \phi^{\rightarrow}_{\mathcal{F}}(g,\beta)$,
since $(f \circ \phi, \alpha) \preccurlyeq (g \circ \phi, \beta)$.

(FF2) $\phi^{\rightarrow}_{\mathcal{F}}(f,\alpha) \otimes \phi^{\rightarrow}_{\mathcal{F}}(g,\beta) = \mathcal{F}(f \circ \phi, \alpha) \otimes \mathcal{F}(g \circ \phi, \beta) \leqslant \mathcal{F}((f \otimes g) \circ \phi, \alpha \vee \beta) = \phi^{\rightarrow}_{\mathcal{F}}(f \otimes g, \alpha \vee \beta)$,
since $(f \circ \phi) \otimes (g \circ \phi) = [(f \otimes g) \circ \phi]$.

(FF3) $\phi^{\rightarrow}_{\mathcal{F}}(0_Y,\alpha) = \mathcal{F}(0_Y \circ \phi, \alpha) = \mathcal{F}(0_X,\alpha) = \bot, \quad \forall \alpha \in L.$

(2). Let $\mathcal{U} : L^X \times L \to L$ be an L-fuzzy ultrafilter on X, let $(g, \beta) \in L^Y \times L$ and let $\alpha \in L$, then

$$\begin{aligned}
\phi^{\rightarrow}_{\mathcal{U}}(g, \beta) &= \mathcal{U}(g \circ \phi, \beta) \\
&= \mathcal{U}\left[(g \circ \phi, \beta) \longmapsto (0_X, \alpha)\right] \longmapsto \perp \\
&= \mathcal{U}[(g \circ \phi) \longmapsto 0_X, \alpha \rhd \beta] \longmapsto \perp \\
&= \mathcal{U}[(g \circ \phi) \longmapsto (0_Y \circ \phi), \alpha \rhd \beta] \longmapsto \perp \\
&= \mathcal{U}[(g \longmapsto 0_Y) \circ \phi, \alpha \rhd \beta] \longmapsto \perp \\
&= \phi^{\rightarrow}_{\mathcal{U}}[g \longmapsto 0_Y, \alpha \rhd \beta] \longmapsto \perp \\
&= \phi^{\rightarrow}_{\mathcal{U}}[(g, \beta) \longmapsto (0_Y, \alpha)] \longmapsto \perp
\end{aligned}$$

.

We conclude from 2.9 that $\phi^{\rightarrow}_{\mathcal{U}} : L^Y \times L \to L$ is an L-fuzzy ultrafilter on Y. □

Proposition 2.11. *Let $\phi : X \to Y$ be a surjective map and let $\mathcal{F} : L^Y \times L \to L$ be an L-fuzzy filter on Y. Then the map $\phi^{\leftarrow}_{\mathcal{F}} : L^X \times L \to L$ defined by $\phi^{\leftarrow}_{\mathcal{F}}(f, \alpha) = \bigvee \{\mathcal{F}(g, \beta) \mid (g \circ \phi, \beta) \preccurlyeq (f, \alpha)\}, \quad \forall (f, \alpha) \in L^X \times L,$ is an L-fuzzy filter on X.*

Proof. We must show $\phi^{\leftarrow}_{\mathcal{F}}$ satisfies the axioms of an L-fuzzy filter:

(FF0)

$$\begin{aligned}
\phi^{\leftarrow}_{\mathcal{F}}(1_X, \alpha) &= \bigvee \{\mathcal{F}(g, \beta) \mid (g \circ \phi, \beta) \preccurlyeq (1_X, \alpha)\} \\
&= \bigvee_{g \in L^Y} \bigwedge_{\beta \in L} \{\mathcal{F}(g, \beta) \mid g \circ \phi \leqslant 1_X, \;\; \alpha \leqslant \beta\} \\
&= \mathcal{F}(1_Y, \alpha) \\
&= \top, \quad \forall \alpha \in L.
\end{aligned}$$

(FF1) $(f,\alpha) \preccurlyeq (g,\beta)$ implies

$$\begin{aligned}
\phi^{\leftarrow}_{\mathcal{F}}(f,\alpha) &= \bigvee \{\mathcal{F}(h,\delta) \mid (h\circ\phi,\delta) \preccurlyeq (f,\alpha)\} \\
&= \bigvee_{h\in L^Y} \bigwedge_{\delta\in L} \{\mathcal{F}(h,\delta) \mid h\circ\phi \leqslant f,\ \ \alpha \leqslant \delta\} \\
&\leqslant \bigvee_{h\in L^Y} \bigwedge_{\delta\in L} \{\mathcal{F}(h,\delta) \mid h\circ\phi \leqslant g,\ \ \beta \leqslant \delta\} \\
&= \bigvee \{\mathcal{F}(h,\delta) \mid (h\circ\phi,\delta) \preccurlyeq (g,\beta)\} \\
&= \phi^{\leftarrow}_{\mathcal{F}}(g\,\beta).
\end{aligned}$$

(FF2)

$$\begin{aligned}
&\phi^{\leftarrow}_{\mathcal{F}}(f,\alpha) \otimes \phi^{\leftarrow}_{\mathcal{F}}(g,\beta) = \\
&= \bigvee \{\mathcal{F}(h,\delta) \mid (h\circ\phi,\delta) \preccurlyeq (f,\alpha)\} \otimes \bigvee \{\mathcal{F}(j,\eta) \mid (j\circ\phi,\eta) \preccurlyeq (g,\beta)\} \\
&\leqslant \bigvee \{\mathcal{F}(h,\delta) \otimes F(j,\eta) \mid ((h\otimes j)\circ\phi, \delta\vee\eta) \preccurlyeq (f\otimes g, \alpha\vee\beta)\} \\
&\leqslant \bigvee \{\mathcal{F}(h\otimes j, \delta\vee\eta) \mid ((h\otimes j)\circ\phi, \delta\vee\eta) \preccurlyeq (f\otimes g, \alpha\vee\beta)\} \\
&= \phi^{\leftarrow}_{\mathcal{F}}(f\otimes g, \alpha\vee\beta).
\end{aligned}$$

(FF3)

$$\begin{aligned}
\phi^{\leftarrow}_{\mathcal{F}}(0_X,\alpha) &= \bigvee \{\mathcal{F}(h,\delta) \mid (h\circ\phi,\delta) \preccurlyeq (0_X,\alpha)\} \\
&= \bigvee_{h\in L^Y} \bigwedge_{\delta\in L} \{\mathcal{F}(h,\delta) \mid h\circ\phi \leqslant 0_X,\ \ \alpha \leqslant \delta\} \\
&= \mathcal{F}(0_Y,\alpha), \quad \text{since } \phi \text{ is surjective} \\
&= \perp.
\end{aligned}$$

□

Definition 2.12 (**L-fuzzy neighborhood system**). *Let X be a set. A map $\mathcal{N} : X \to L^{(L^X\times L)}$ is called an L-fuzzy neighborhood system on X if and only if, for each $p \in X$, the mapping $\mathcal{N}_p : L^X \times L \to L$ satisfies the following axioms:*

(N_0) $\mathcal{N}_p(1_X, \alpha) = \top$.

(N_1) $(f, \alpha) \preccurlyeq (g, \beta) \Rightarrow \mathcal{N}_p(f, \alpha) \leqslant \mathcal{N}_p(g, \beta)$.

(N_2) $\mathcal{N}_p(f, \alpha) \otimes \mathcal{N}_p(g, \beta) \leqslant \mathcal{N}_p(f \otimes g, \alpha \vee \beta)$.

(N_3) $\mathcal{N}_p(f, \alpha) \leqslant f(p)$.

(N_4) $\mathcal{N}_p(f, \alpha) \leqslant \bigvee \{\mathcal{N}_p(g, \beta) \mid (f, \alpha) \preccurlyeq (g, \beta) \text{ and } g(q) \leqslant \mathcal{N}_q(f, \alpha), \ \forall q \in X\}$.

Definition 2.13 (**L-fuzzy interior operator**). *Let X be a set. A map $\mathcal{I} : L^X \times L \to L^X$ is called an L-fuzzy interior operator on X if and only if $\mathcal{I}$ satisfies the following conditions:*

(I_0) $\mathcal{I}(1_X, \alpha) = 1_X, \quad \forall \alpha \in L$.

(I_1) $(f, \alpha) \preccurlyeq (g, \beta) \Rightarrow \mathcal{I}(f, \alpha) \leqslant \mathcal{I}(g, \beta)$.

(I_2) $\mathcal{I}(f, \alpha) \otimes \mathcal{I}(g, \beta) \leqslant \mathcal{I}(f \otimes g, \alpha \vee \beta)$.

(I_3) $\mathcal{I}(f, \alpha) \leqslant f$.

(I_4) $\mathcal{I}(f, \alpha) \leqslant \mathcal{I}\big(\mathcal{I}(f, \alpha)\big)$.

(I_5) $\mathcal{I}(f, \perp) = f$.

(I_6) *If* $\emptyset \neq K \subseteq L$, $\mathcal{I}(f, \alpha) = f^0 \ \ \forall \alpha \in K$, *then* $I(f, \bigvee K) = f^0$.

Lemma 2.14. *(cf. [9]) Given an L-fuzzy topology $\mathcal{T} : L^X \to L$ on a set X, the mapping $\mathcal{I}_\mathcal{T} : L^X \times L \to L^X$ defined by*

$$\mathcal{I}(f, \alpha) := \mathcal{I}_\mathcal{T}(f, \alpha) = \bigvee \{u \in L^X \mid (u, \mathcal{T}(u)) \preccurlyeq (f, \alpha)\}, \quad \forall (f, \alpha) \in L^X \times L$$

is an L-fuzzy interior operator on X.

Lemma 2.15. *(cf. [9]) Every L-fuzzy interior operator $\mathcal{I} : L^X \times L \to L^X$ induces, for each $p \in X$, an L-fuzzy neighborhood system $\mathcal{N}_p^\mathcal{I} : L^X \times L \to L$ defined by*

$$\mathcal{N}_p(f, \alpha) := \mathcal{N}_p^\mathcal{I}(f, \alpha) = [\mathcal{I}(f, \alpha)](p).$$

Proposition 2.16. *Let $(X, \mathcal{T})$ and (Y, σ) be a pair of L-fuzzy topological spaces, let $\phi : X \to Y$ be an L-fuzzy continuous surjective map, let*

$\mathcal{N}_p : L^X \times L \to L$ be the L-fuzzy neighborhood system of a point $p \in X$ induced by $\mathcal{T}$, and let $\mathcal{N}_{\phi(p)} : L^Y \times L \to L$ be the corresponding L-fuzzy neighborhood system of $\phi(p)$ in Y induced by σ. Then

$$\mathcal{N}_{\phi(p)} \leqslant \phi^{\to}(\mathcal{N}_p)$$

where $\phi^{\to}(\mathcal{N}_p) : L^Y \times L \to L$ is defined by

$$\phi^{\to}(\mathcal{N}_p)(g, \beta) = \mathcal{N}_p(g \circ \phi, \beta), \quad \forall (g, \beta) \in L^Y \times L.$$

Proof. From lemma 2.14 we have that

$$\mathcal{I}_\sigma(g, \beta) = \bigvee \{u \in L^Y \mid (u, \sigma(u)) \preccurlyeq (g, \beta)\}, \quad \forall (g, \beta) \in L^Y \times L$$

is an L-fuzzy interior operator on X. The L-fuzzy continuity of ϕ implies that

$$\sigma(u) \leqslant \mathcal{T}(u \circ \phi), \qquad \forall u \in L^Y.$$

We therefore have that

$$\{u \in L^Y \mid u \leqslant g, \;\; \beta \leqslant \sigma(u)\} \subseteq \{v \in L^Y \mid v \leqslant g, \;\; \beta \leqslant \mathcal{T}(v \circ \phi)\},$$

which implies

$$\bigvee \{u \in L^Y \mid u \leqslant g, \;\; \beta \leqslant \sigma(u)\} \leqslant \bigvee \{v \in L^Y \mid v \leqslant g, \;\; \beta \leqslant \mathcal{T}(v \circ \phi)\}.$$

In other words,

$$\mathcal{I}_\sigma(g, \beta) \leqslant \bigvee \{v \in L^Y \mid v \leqslant g, \;\; \beta \leqslant \mathcal{T}(v \circ \phi)\}.$$

If $\omega = \bigvee \{v \in L^Y \mid v \leqslant g, \;\; \beta \leqslant \mathcal{T}(v \circ \phi)\}$, then

$$\mathcal{N}_{\phi(p)}(g, \beta) = [\mathcal{I}_\sigma(g, \beta)] \, (\phi(p)) \leqslant \omega \, (\phi(p)) = (\omega \circ \phi) \, (p),$$

and so

(2.16.1) $$\mathcal{N}_{\phi(p)}(g, \beta) \leqslant (\omega \circ \phi) \, (p).$$

On the other hand, it follows from $\omega \circ \phi \leqslant g \circ \phi$, and

$$\mathcal{N}_p(g \circ \phi, \beta) = [\mathcal{I}_\mathcal{T}(g \circ \phi, \beta)] \, (p) = \bigvee \{u \in L^X \mid u \leqslant g \circ \phi, \;\; \beta \leqslant \mathcal{T}(u)\}(p),$$

and $\omega \circ \phi \in \{u \in L^X \mid u \leqslant g \circ \phi,\ \ \beta \leqslant \mathcal{T}(u)\}$, that

$$(\omega \circ \phi)(p) \leqslant \bigvee \{u \in L^X \mid u \leqslant g \circ \phi,\ \ \beta \leqslant \mathcal{T}(u)\}(p)$$
$$= [\mathcal{I}_{\mathcal{T}}(g \circ \phi, \beta)]\,(p) = \mathcal{N}_p(g \circ \phi, \beta) = \phi^{\rightarrow}_{(\mathcal{N}_p)}(g, \beta),$$

and so

(2.16.2) $$(\omega \circ \phi)(p) \leqslant \phi^{\rightarrow}_{(\mathcal{N}_p)}(g, \beta).$$

Finally, from 2.16.1 and 2.16.2, we conclude that

$$\mathcal{N}_{\phi(p)} \leqslant \phi^{\rightarrow}(\mathcal{N}_p).$$

□

Definition 2.17 (Adherent point). *Let (X, τ) be an L-fuzzy topological space and $\mathcal{N} : X \to L^{(L^X \times L)}$ the corresponding L-fuzzy neighborhood system. Further, let $\mathcal{F} : L^X \times L \to L$ be an L-fuzzy filter on X. A point $p \in X$ is called an adherent point of $\mathcal{F}$ iff there exists a further L-fuzzy filter $\mathcal{G}$ on X provided with the following properties*

(i) $\mathcal{G}(\perp \otimes \top).1_X, \alpha) \leqslant \perp \otimes \top, \quad \forall \alpha \in L.$

(ii) $\mathcal{N}_p \leqslant \mathcal{G}$ *and* $\mathcal{F} \leqslant \mathcal{G}$.

Definition 2.18. *An L-fuzzy topological space (X, τ) is compact iff each L-fuzzy filter on X has at least one adherent point.*

Proposition 2.19. *Let (X, τ) be a compact L-fuzzy topological space, let (Y, σ) be an L-fuzzy topological space and let $\phi : X \to Y$ be a surjective, L-fuzzy continuous map. Then (Y, σ) is compact.*

Proof. Let $\mathcal{F} : L^Y \times L \to L$ be an L-fuzzy filter on Y, then by 2.11, $\phi^{\leftarrow}_{\mathcal{F}}$ is an L-fuzzy filter on X. Since (X, τ) is compact, $\phi^{\leftarrow}_{\mathcal{F}}$ has an adherent point $p \in X$, i.e. there exists an L-fuzzy filter $\mathcal{G}$ on X with $\mathcal{N}_p \leqslant \mathcal{G}$ and $\phi^{\leftarrow}_{\mathcal{F}} \leqslant \mathcal{G}$.

Now we form, using 2.10, the corresponding L-fuzzy filter image and we obtain from the surjectivity of ϕ the following relations

$$\mathcal{F} = \phi^{\rightarrow}\left(\phi^{\leftarrow}(\mathcal{F})\right) \leqslant \phi^{\rightarrow}(\mathcal{G}) \quad \text{and} \quad \phi^{\rightarrow}(\mathcal{N}_p) \leqslant \phi^{\rightarrow}(\mathcal{G}).$$

On the other hand, from 2.16 follows

$$\mathcal{N}_{\phi(p)} \leqslant \phi^{\rightarrow}(\mathcal{N}_p),$$

which implies that

$$\mathcal{F} \leqslant \phi^{\rightarrow}(\mathcal{G}) \quad \text{and} \quad \mathcal{N}_{\phi(p)} \leqslant \phi^{\rightarrow}(\mathcal{G}),$$

hence $\phi(p)$ is an adherent point of $\mathcal{F}$.

This completes the proof of the proposition. □

In order to get a proof of our version of Tychonoff's theorem, we need the following lemmas:

Lemma 2.20. *Let $\mathfrak{F} = \{(X_\lambda, \mathcal{T}_\lambda) \mid \lambda \in I\}$ be a non-empty family of L-fuzzy topological spaces and let $(X, \mathcal{T})$ be its L-fuzzy topological product. Then for each $p \in X$, the mapping $\mathcal{N}_p : L^X \times L \to L$ defined by*

$$\mathcal{N}_p(f, \alpha) = \bigvee \left\{ \bigotimes_{\lambda \in I} \mathcal{N}_{p_\lambda}(h_\lambda, \alpha) \mid h \in \Gamma_f,\ \ \alpha \leqslant \bigotimes_{\lambda \in I} \tau_\lambda(h_\lambda) \right\},$$

where

$$\Gamma_f = \{\mu \in \prod_{\lambda \in I} L^{X_\lambda} \mid \mu_\lambda = 1_{X_\lambda} \textit{for all but finitely many indices } \lambda, \textit{ and } \bigotimes_{\lambda \in I} (\mu_\lambda \circ \pi_\lambda) \leqslant f\},$$

is an L-fuzzy neighborhood system on X.

Proof. We must show that $\mathcal{N}_p$ satisfies the axioms of an L-fuzzy neighborhood system:

(N_0). Let $1_\Delta \in \prod_{\lambda \in I} L^{X_\lambda}$ defined by $(1_\Delta)_\lambda = 1_{X_\alpha}$, for each $\lambda \in I$. Then $1_\Delta \in \Gamma_{1_X}$. Since

(1) $\tau_\lambda(1_{X_\lambda}) = \top$ for all $\lambda \in I$, and so $\bigotimes_{\lambda \in I} \tau_\lambda(1_{X_\lambda}) = \bigotimes_{\lambda \in I} \top = \top$,

(2) $\mathcal{N}_{p_\lambda}(1_{X_\lambda},\alpha)=\top$ for all $\lambda\in I$, and $\bigotimes_{\lambda\in I}\mathcal{N}_{p_\lambda}(1_{X_\lambda},\alpha)=\bigotimes_{\lambda\in I}\top=\top$, then

$$\mathcal{N}_p(1_X,\alpha)=\bigvee\left\{\bigotimes_{\lambda\in I}\mathcal{N}_{p_\lambda}(h_\lambda,\alpha)\mid h\in\Gamma_{1_X},\ \ \alpha\leqslant\bigotimes_{\lambda\in I}\tau_\lambda(h_\lambda)\right\}=\top.$$

We therefore have that $\mathcal{N}_p(1_X,\alpha)=\top$.

(N_1). It is our purpose to show that $\mathcal{N}_p(f,\alpha)\leqslant\mathcal{N}_p(g,\beta)$, whenever $f\leqslant g$ in L^X, and $\beta\leqslant\alpha$ in L.

Clearly $f\leqslant g$ implies that $\Gamma_f\subseteq\Gamma_g$. Consequently,

$$\begin{aligned}&\bigvee\left\{\bigotimes_{\lambda\in I}\mathcal{N}_{p_\lambda}(h_\lambda,\alpha)\mid h\in\Gamma_f,\ \ \alpha\leqslant\bigotimes_{\lambda\in I}\tau_\lambda(h_\lambda)\right\}\\ \leqslant&\bigvee\left\{\bigotimes_{\lambda\in I}\mathcal{N}_{p_\lambda}(h_\lambda,\beta)\mid h\in\Gamma_g,\ \ \beta\leqslant\bigotimes_{\lambda\in I}\tau_\lambda(h_\lambda)\right\}.\end{aligned}$$

This verifies that $\mathcal{N}_p(f,\alpha)\leqslant\mathcal{N}_p(g,\beta)$.

(N_2). We must show that $\mathcal{N}_p(f,\alpha)\otimes\mathcal{N}_p(g,\beta)\leqslant\mathcal{N}_p(f\otimes g,\alpha\vee\beta)$.

Since $\mathcal{N}_p(f,\alpha)=\bigvee\left\{\bigotimes_{\lambda\in I}\mathcal{N}_{p_\lambda}(h_\lambda,\alpha)\mid h\in\Gamma_f,\ \ \alpha\leqslant\bigotimes_{\lambda\in I}\tau_\lambda(h_\lambda)\right\}$ and $\mathcal{N}_p(g,\beta)=\bigvee\left\{\bigotimes_{\lambda\in I}\mathcal{N}_{p_\lambda}(j_\lambda,\beta)\mid j\in\Gamma_g,\ \ \beta\leqslant\bigotimes_{\lambda\in I}\tau_\lambda(j_\lambda)\right\}$, and since $\otimes$ is distributive over non-empty joins, we have that

$$\begin{aligned}&\mathcal{N}_p(f,\alpha)\otimes\mathcal{N}_p(g,\beta)\\ &=\bigvee\left\{\bigotimes_{\lambda\in I}\mathcal{N}_{p_\lambda}(h_\lambda,\alpha)\otimes\bigotimes_{\lambda\in I}\mathcal{N}_{p_\lambda}(j_\lambda,\beta)\mid h\in\Gamma_f,j\in\Gamma_g,\ \alpha\vee\beta\leqslant\bigotimes_{\lambda\in I}\tau_\lambda(h_\lambda\otimes j_\lambda)\right\}\\ &=\bigvee\left\{\bigotimes_{\lambda\in I}[\mathcal{N}_{p_\lambda}(h_\lambda,\alpha)\otimes\mathcal{N}_{p_\lambda}(j_\lambda,\beta)]\mid h\in\Gamma_f,j\in\Gamma_g,\ \alpha\vee\beta\leqslant\bigotimes_{\lambda\in I}\tau_\lambda(h_\lambda\otimes j_\lambda)\right\}\\ &\leqslant\bigvee\left\{\bigotimes_{\lambda\in I}\mathcal{N}_{p_\lambda}(h_\lambda\otimes j_\lambda,\alpha\vee\beta)\mid h\otimes j\in\Gamma_{f\otimes g},\ \alpha\vee\beta\leqslant\bigotimes_{\lambda\in I}\tau_\lambda(h_\lambda\otimes j_\lambda)\right\}\\ &=\mathcal{N}_p(f\otimes g,\alpha\vee\beta).\end{aligned}$$

(N_3). We must verify that $N_p(f,\alpha)\leqslant f(p)$, for each $(f,\alpha)\in L^X\times L$.

Since $\mathcal{N}_{p_\lambda}$ is an L-fuzzy neighborhood system on X_λ, we have that $\mathcal{N}_{p_\lambda}(h_\lambda,\alpha)\leqslant h_\lambda(p_\lambda)$, for all $\lambda\in I$, and for all $(h_\lambda,\alpha)\in L^{X_\lambda}\times L$. Now,

let $h \in \Gamma_f$, then $h \in \prod_{\lambda \in I} L^{X_\lambda}$, $h_\lambda = 1_{X_\lambda}$ for all but finitely many indices $\lambda \in I$, and $\bigotimes_{\lambda \in I}(h_\lambda \circ \pi_\lambda) \leqslant f$. Hence

$$\bigotimes_{\lambda \in I}(h_\lambda \circ \pi_\lambda)(p) = \bigotimes_{\lambda \in I}(h_\lambda)(p_\lambda) \leqslant f(p).$$

We therefore have that

$$\bigotimes_{\lambda \in I} \mathcal{N}_{p_\lambda}(h_\lambda, \alpha) \leqslant \bigotimes_{\lambda \in I}(h_\lambda)(p_\lambda) \leqslant f(p),$$

which implies

$$\mathcal{N}_p(f, \alpha) = \bigvee \left\{ \bigotimes_{\lambda \in I} \mathcal{N}_{p_\lambda}(h_\lambda, \alpha) \mid h \in \Gamma_f,\ \ \alpha \leqslant \bigotimes_{\lambda \in I} \tau_\lambda(h_\lambda) \right\} \leqslant f(p).$$

(N_4). To show that

$$\mathcal{N}_p(f, \alpha) \leqslant \bigvee \left\{ \mathcal{N}_p(g, \beta) \mid (f, \alpha) \preccurlyeq (g, \beta),\ g(q) \leqslant \mathcal{N}_q(f, \alpha),\ \forall q \in X \right\},$$

we must verify the following:

$$\mathcal{N}_{p_\lambda}(\omega_\lambda, \alpha) \leqslant \bigvee \left\{ \mathcal{N}_{p_\lambda}(v_\lambda, \beta) \mid (\omega_\lambda, \alpha) \preccurlyeq (v_\lambda, \beta),\ v_\lambda(q_\lambda) \leqslant \mathcal{N}_{q_\lambda}(\omega_\lambda, \alpha),\ \forall q_\lambda \in X_\lambda \right\}$$

implies

$$\mathcal{N}_p(f, \alpha) \leqslant \bigvee \left\{ \mathcal{N}_p(g, \beta) \mid (f, \alpha) \preccurlyeq (g, \beta),\ g(q) \leqslant \mathcal{N}_q(f, \alpha),\ \forall q \in X \right\}.$$

We observe from the hypothesis that, for $\omega \in \Gamma_f$ and $v \in \Gamma_g$,

$$\begin{aligned}
&\bigotimes_{\lambda \in I} \mathcal{N}_{p_\lambda}(\omega_\lambda, \alpha) \\
&\leqslant \bigvee \left\{ \bigotimes_{\lambda \in I} \mathcal{N}_{p_\lambda}(v_\lambda, \beta) \mid (\omega_\lambda, \alpha) \preccurlyeq (v_\lambda, \beta),\ v_\lambda(q_\lambda) \leqslant \mathcal{N}_{q_\lambda}(\omega_\lambda, \alpha),\ \forall q_\lambda \in X_\lambda \right\} \\
&\leqslant \bigvee \left\{ \bigotimes_{\lambda \in I} \mathcal{N}_{p_\lambda}(v_\lambda \circ \pi_\lambda, \beta) \mid (\omega_\lambda, \alpha) \preccurlyeq (v_\lambda, \beta),\ v_\lambda(q_\lambda) \leqslant \mathcal{N}_{q_\lambda}(\omega_\lambda, \alpha),\ \forall q_\lambda \in X_\lambda \right\}.
\end{aligned}$$

We therefore have that

$$\mathcal{N}_p(f, \alpha) \leqslant \bigvee \left\{ \mathcal{N}_p(g, \beta) \mid (f, \alpha) \preccurlyeq (g, \beta),\ g(q) \leqslant \mathcal{N}_q(f, \alpha),\ \forall q \in X \right\}.$$

□

*****Lemma 2.21.*** *Let $\mathfrak{F} = \{(X_\lambda, \mathcal{T}_\lambda) \mid \lambda \in I\}$ be a non-empty family of compact L-fuzzy topological spaces and let $(X, \mathcal{T})$ be its L-fuzzy topological product. Let $\mathcal{U} : L^X \times L \to L$ be an L-fuzzy ultrafilter on X, then the following statements are equivalent:*

(1) *$\mathcal{U}$ converges to an element $p = (p_\lambda)_{\lambda \in I}$ in X.*

(2) *For each $\lambda \in I$, $\pi_\lambda^{\rightarrow}(\mathcal{U})$ converges to p_λ, where $\pi_\lambda; X \to X_\lambda$ is the λth projection.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1). Suppose that $\pi_\lambda^{\rightarrow}(\mathcal{U})$ converges to p_λ, for each $\lambda \in I$. Then $\mathcal{N}_{p_\lambda} \leqslant \pi_\lambda^{\rightarrow}(\mathcal{U})$. Therefore

$$\begin{aligned}
\bigotimes_{\lambda \in I} \mathcal{N}_{p_\lambda}(h_\lambda, \alpha) &\leqslant \bigotimes_{\lambda \in I} \pi_\lambda^{\rightarrow}(\mathcal{U})(h_\lambda, \alpha) \\
&= \bigotimes_{\lambda \in I} \mathcal{U}(h_\lambda \circ \pi_\lambda, \alpha) \\
&\leqslant \mathcal{U}\left(\bigotimes_{\lambda \in I} (h_\lambda \circ \pi_\lambda), \alpha \right) \\
&\leqslant \mathcal{U}\left(f, \alpha\right).
\end{aligned}$$

It follows that

$$\mathcal{N}_p(f, \alpha) = \bigvee \left\{ \bigotimes_{\lambda \in I} \mathcal{N}_{p_\lambda}(h_\lambda, \alpha) \mid h \in \Gamma_f, \;\; \alpha \leqslant \bigotimes_{\lambda \in I} \tau_\lambda(h_\lambda) \right\} \leqslant \mathcal{U}\left(f, \alpha\right).$$

This establishes that $\mathcal{U}$ converges to an element $p = (p_\lambda)_{\lambda \in I}$ in X. □

As a consequence of the previous lemmas, we get the main result:

***Theorem 2.22.** *Let $\mathfrak{F} = \{(X_\lambda, \mathcal{T}_\lambda) \mid \lambda \in I\}$ be a non-empty family of L-fuzzy topological spaces. Then the following assertions are equivalent:*

(1) *$(X_\lambda, \mathcal{T}_\lambda)$ is compact, for all $\lambda \in I$.*

(2) *The L-fuzzy topological product $(X_0, \mathcal{T}_0)$ of $\mathfrak{F}$ in the category* **L-FTop** *is compact.*

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